Introduction

Intuitively, self-similar spaces are the spaces that resemble parts of themselves. This is a property found in fractals, but also to more familiar spaces such as the real line. It turns out that in a complete metric space $X$ (such as the plane), a set of contraction maps $\mathcal{S} = \{S_1, \ldots, S_k\}$ uniquely defines a self-similar space in $X$. Given such a set of maps, Hutchinson [3] showed there is a unique closed bounded $K \subseteq X$ such that:

$$ K = \bigcup_{i=1}^{N} S_i(K) $$

We say $K$ is invariant w.r.t $\mathcal{S}$ i.e. when each map is applied to $K$, and the results glued together, we obtain the original space itself.

Why study self-similar spaces? They are interesting from a theoretical point of view, but do they have a more practical use? Although self-similar spaces seem abstract, they occur in nature everywhere. The paradigm of recursively applying basic rules to create vast complex shapes is seen everywhere, such as in the formation of ferns and the growth of crystals (see Figure 1.3).

![Fig 1.1. The Sierpinski gasket.](image)

![Fig 1.2. A Mandelbrot fractal.](image)

![Fig 1.3. A fern fractal.](image)

Representing Self-Similar Spaces

Computationally, we can represent self-similar spaces using a coordinate map $\pi : C(N) \rightarrow K$, where $C(N)$ is the cantor set over $N$ elements i.e. infinite strings of numbers between 1 and $N$. $\pi$ is defined below (see Figure 2 for an illustration):

$$ \pi(a) = \frac{\sum_{i=1}^{N} S_{a_i}(K)}{K} $$

Indeed, any function $f : K \rightarrow J$ between self-similar spaces can be realised as a continuous $f : C(N) \rightarrow C(M)$ such that the following diagram commutes:

$$ C(N) \xrightarrow{f} C(M) $$

$$ \xrightarrow{\pi} \frac{J}{J} $$

For example, the unit interval $[0,1]$ is the invariant space arising from the set of contractions $\mathcal{N} = \{S_1 : x \mapsto \frac{x}{2}, S_2 : x \mapsto \frac{1-x}{2}\}$. The corresponding representation is $\pi : C(2) \rightarrow [0,1]$ which, when unravelled, gives the standard binary representation:

$$ \pi(a_0a_1\ldots) = S_{a_0} \circ \cdots \circ S_{a_2} \circ \cdots \circ S_{a_1} $$

$$ = \frac{a_0}{2} + \frac{a_1}{4} + \frac{a_2}{8} + \cdots $$

There are many other possible representations for real numbers such as continued fractions, Dedekind cuts, Cauchy sequences, linear fractional transforms etc. All have their particular uses.

Measures on Self-Similar Spaces

Measures generalise the concept of volume and area to arbitrary spaces and (as a consequence) also generalise integration. Informally, a measure on a set $X$ is a map $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ which assigns to every $A \in \mathcal{P}(X)$, a positive real number representing its measure/volume/area/colour/probability, depending on the context we are working in.

Just as the set $\mathcal{S}$ determines $K$, it also determines a measure on $K$. By specifying a discrete measure on the set $\{1, \ldots, N\}$, we induce a measure on $C(N)$ and thus on $K$ via the coordinate map $\pi$ (see Figure 3 for an illustration):

$$ \mu(A) = \sum_{i=1}^{N} \rho_i \cdot \mu(S_i(A)) $$

$$ \int_a^b f \, d\mu = \sum_{i=1}^{N} \rho_i \cdot \int_a^b f \, d\mu $$

In our case the discrete measure on $\{1, \ldots, N\}$ is simply a set of real numbers $\rho = \{\rho_1, \ldots, \rho_N\}$ such that $\rho_i \in (0,1)$ and $\sum \rho_i = 1$. Intuitively, we think of $\rho_i$ as the weighting associated with each component, $S_i(K)$, of $K$.

"Self-similarity occurs everywhere in nature: by combining simple rules we create vastly complex spaces with amazing latent structure."

The recursive nature of $\mu$ allows us to construct an algorithm for integration over $K$, inspired from [4]:

$$ k\text{-int} :: (K \rightarrow I) \rightarrow I $$

$$ k\text{-int} (f) = \text{let } d = \text{head}(f([1..])) \text{ in } $$

$$ \text{if forall ( } \forall v \rightarrow (\text{head}.f(v) == d) \text{ ) then } d : k\text{-int}(\text{tail}.f) \text{ else w-sum}(n,p,\i \rightarrow k\text{-int}(f.S(i))) $$

Here $K$ is a representation of some self-similar space, and $I$ is a representation of the unit interval $[0,1]$. $\forall x$ is the universal quantifier over $K$, an amazing algorithm due to Berger [1], and $\text{w-sum}$ is a program for the weighted sum:

$$ \text{w-sum}(a,p,f) = \sum_{i=1}^{a} [p[i]] [f(i)] $$

The idea behind the algorithm is to repeatedly apply (1), branching until we achieve a function that is "constant" enough for us to extract a digit. The process is illustrated in Figure 4.

![Fig 3. An illustration of the invariant measure on the Sierpinski gasket with coefficients $\rho = (\frac{1}{2}, \frac{1}{4})$. The shade of grey represents the measure of that subset. By (1) the set of sequences in $C(N)$ starting 231... has measure $\mu(231\ldots) = \rho_2\rho_3 = \frac{1}{8}$.](image)

![Fig 4. The integration program for $K = [0,1]$.](image)

Reference


